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Algebraic-geometrical solutions of some multidimensional nonlinear evolution equations

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Abstract
The known (2+1)-dimensional breaking soliton equation, the coupled KP equation with three potentials and a new (3+1)-dimensional nonlinear evolution equation are decomposed into systems of solvable ordinary differential equations with the help of the (1+1)-dimensional AKNS equations. The Abel–Jacobi coordinates are introduced to straighten out the associated flows, from which algebraic-geometrical solutions of the (2+1)-dimensional breaking soliton equation, the coupled KP equation and the (3+1)-dimensional evolution equation are explicitly given in terms of the Riemann theta functions.

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1. Introduction

The construction of explicit solutions for multidimensional soliton equations is an important task. However, it is very difficult to solve them due to their multispatial dimensions and nonlinearity. Usually one considers the multidimensional problems to be solved in such a way as splitting into several lower-dimensional ones, which are more easily treated with some available tools. The nonlinearization approach of Lax pairs [1–3] makes it possible to decompose the (1+1)-dimensional soliton equations into the compatible ordinary differential equations, which are the finite-dimensional completely integrable systems in the Liouville sense [1–4]. The (2+1)-dimensional soliton equations could be decomposed in a similar procedure from their Lax representation into the (1+1)-dimensional soliton systems [5–8], and further into the compatible ordinary differential equations. This paves a way of solving the (1+1)- and (2+1)-dimensional soliton equations.

Algebraic-geometrical solutions, called also quasi-periodic solutions, of soliton equations are important, which can be used to find multisoliton solutions through the degeneracy procedure [9]. Various methods have been developed to get algebraic-geometrical solutions of (1+1)-dimensional soliton equations, for instance, the algebraic-geometrical approach (see, e.g., [10] and references therein), and others [11–20]. However, studies of algebraic-geometrical solutions for (2+1)-dimensional soliton equations are very few. Recently, algebraic-geometrical solutions of some (2+1)-dimensional soliton equations such as the Kadometsev–Petviashvili (KP), the mKP, the special (2+1)-dimensional Toda lattice, the (2+1)-dimensional Gardner equations have been successfully obtained resorting to the nonlinearization of Lax pairs and finite-order expansion of the Lax matrix [21–24].

The aim of the present paper is to study the decomposition of three multidimensional nonlinear evolution equations and the construction of their algebraic-geometrical solutions. These nonlinear evolution equations are the (2+1)-dimensional breaking soliton equation [25, 26]

$$\hat{q}_t = \hat{q}_{xy} - 2\hat{q}\partial_x^{-1}(\hat{q}\hat{r})_y \qquad \hat{r}_t = -\hat{r}_{xy} + 2\hat{r}\partial_x^{-1}(\hat{q}\hat{r})_y \tag{1.1}$$

the coupled KP equation with three potentials [28–30]

$$\begin{aligned} q_t &= \tfrac{1}{4}\left(q_{xxx} - 6qq_x + 3\partial_x^{-1}q_{yy} + 6(pr)_x\right) \\ p_t &= \tfrac{1}{2}\left(-p_{xxx} + 3qp_x - 3p_{xy} + 3p\partial_x^{-1}q_y\right) \\ r_t &= \tfrac{1}{2}\left(-r_{xxx} + 3qr_x + 3r_{xy} - 3r\partial_x^{-1}q_y\right) \end{aligned} \tag{1.2}$$

and a new (3+1)-dimensional nonlinear evolution equation

$$3w_{xz} - (2w_t + w_{xxx} - 2ww_x)_y + 2\big(w_x\partial_x^{-1}w_y\big)_x = 0 \tag{1.3}$$

where ∂_x^{-1} stands for an inverse operator of $\partial_x = \partial/\partial x$ with the condition $\partial_x\partial_x^{-1} = \partial_x^{-1}\partial_x = 1$, which can be defined as $\big(\partial_x^{-1}f\big)(x) = \int_{-\infty}^{x} f(x')\,\mathrm{d}x'$ under the decaying condition at infinity. Equation (1.1) was studied in a series of papers [25–27] and was used to describe the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis. This equation can be solved via the inverse scattering method. It has been shown that equation (1.1) possesses the Hamiltonian structure and infinitely many symmetries. And these symmetries usually constitute some infinite-dimensional Lie algebras (see, e.g., [27] and references therein). For the coupled KP equation (1.2), the N-soliton solution, the bilinear form and other systematic results were obtained in [28–30].

In this paper, based on the known (1+1)-dimensional AKNS equations it is shown that solutions of the (2+1)-dimensional breaking soliton equation (1.1), the coupled KP equation (1.2) and the (3+1)-dimensional nonlinear evolution equation (1.3) are reduced to solvable ordinary differential equations, from which algebraic-geometrical solutions of the (2+1)-dimensional breaking soliton equation (1.1), the coupled KP equations (1.2) and the (3+1)-dimensional nonlinear evolution equation (1.3) are obtained. The present paper is organized as follows. In section 2, we shall decompose the (2+1)-dimensional breaking soliton equation (1.1), the coupled KP equation (1.2) and the (3+1)-dimensional nonlinear evolution equation (1.3) into the first two or three members of the AKNS hierarchy in a direct way and the nonlinearization of a Lax pair. Here a Lax pair of the KP equation (1.2) is proposed. In section 3, with the help of solutions for the (1+1)-dimensional stationary AKNS equations, we introduce the elliptic coordinates, by which solutions of the AKNS hierarchy, the (2+1)-dimensional breaking soliton equation (1.1), the coupled KP equation (1.2) and the (3+1)-dimensional evolution equations (1.3) are reduced to solving systems of solvable ordinary differential equations. In section 4, a hyperelliptic Riemann surface of genus N and Abel–Jacobi coordinates are defined to straighten out the associated flows. The Jacobi inversion problem is discussed, from which algebraic-geometrical solutions of the (2+1)-dimensional breaking soliton equation (1.1), the coupled KP equation (1.2) and the (3+1)-dimensional nonlinear evolution equation (1.3) are expressed explicitly in terms of the Riemann theta functions.

2. Decomposition of multidimensional evolution equations

In this section, we first decompose the (2+1)-dimensional breaking soliton equation (1.1), the coupled KP equation (1.2) and the (3+1)-dimensional nonlinear evolution equation (1.3) into the (1+1)-dimensional AKNS equations. To this end, we consider the first three members of the AKNS hierarchy [16, 17, 31]:

$$u_y = -u_{xx} + 2u^2 v \qquad v_y = v_{xx} - 2uv^2 \tag{2.1}$$

$$u_t = u_{xxx} - 6uvu_x \qquad v_t = v_{xxx} - 6uvv_x \tag{2.2}$$

and

$$\begin{aligned} u_z &= -u_{xxxx} + 8uvu_{xx} + 6u_x^2 v + 4uu_x v_x + 2u^2 v_{xx} - 6u^3 v^2 \\ v_z &= v_{xxxx} - 8uvv_{xx} - 6uv_x^2 - 4vu_x v_x - 2v^2 u_{xx} + 6u^2 v^3. \end{aligned} \tag{2.3}$$

It is a well-known fact that equations (2.1)–(2.3) are compatible since the flows determined by them commute.

Proposition 2.1. *Let (u, v) be a compatible solution of equations (2.1) and (2.2). Then the functions q and r determined by*

$$\hat{q}(x, y, t) = v \qquad \hat{r}(x, y, t) = u \tag{2.4}$$

solve the (2+1)-dimensional breaking soliton equation (1.1).

Proof. With the aid of (2.1), we have

$$uv_x - vu_x = \partial_x^{-1}(\hat{q}\hat{r})_y. \tag{2.5}$$

Substituting (2.1), that is $u_{xx} = -u_y + 2u^2 v$ and $v_{xx} = v_y + 2uv^2$, into (2.2) and noting (2.5) yield (1.1). □

Proposition 2.2. *The coupled KP equation (1.2) has a Lax pair, which is the spectral problem*

$$\phi_y = U\phi \qquad \phi = \begin{pmatrix} u \\ v \end{pmatrix} \qquad U = \begin{pmatrix} -\partial_x^2 + q + \lambda & p \\ -r & \partial_x^2 - q + \lambda \end{pmatrix} \tag{2.6}$$

and the auxiliary problem

$$\phi_t = V\phi \qquad V = \frac{1}{4}\begin{pmatrix} 4\partial_x^3 - 6q\partial_x - 3q_x + \varsigma & -6p_x \\ -6r_x & 4\partial_x^3 - 6q\partial_x - 3q_x - \varsigma \end{pmatrix} \tag{2.7}$$

where q, p and r are three scalar potentials, λ a constant spectral parameter, $\varsigma = 3\partial_x^{-1}q_y$.

Proof. A direct calculation shows that the compatibility condition of (2.6) and (2.7) yields the Lax equation $U_t - V_y + [U, V] = 0$, which is equivalent to the coupled KP equation (1.2).

As $\lambda = 0$, equations (2.6) and (2.7) can read

$$u_y = -u_{xx} + qu + pv \qquad v_y = v_{xx} - qv - ru \tag{2.8}$$

and

$$\begin{aligned} u_t &= u_{xxx} - \tfrac{3}{2}qu_x - \tfrac{3}{4}q_x u + \tfrac{1}{4}\varsigma u - \tfrac{3}{2}p_x v \\ v_t &= v_{xxx} - \tfrac{3}{2}qv_x - \tfrac{3}{4}q_x v - \tfrac{1}{4}\varsigma v - \tfrac{3}{2}r_x u. \end{aligned} \tag{2.9}$$

Now we impose the constraint between the potentials and eigenfunctions

$$q = 4uv \qquad p = -2u^2 \qquad r = -2v^2 \tag{2.10}$$

which together with (2.8) implies

$$\varsigma = 3\partial_x^{-1} q_y = 12(uv_x - u_x v). \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.8) and (2.9) yields (2.1) and (2.2). Therefore, we obtain the following fact, which can be verified by direct calculations. □

Proposition 2.3. *Let (u, v) be a compatible solution of equations (2.1) and (2.2). Then the function (q, p, r) determined by (2.10) is a solution of the coupled KP equation (1.2).*

Proposition 2.4. *Let (u, v) be a compatible solution of equations (2.1)–(2.3). Then the function w determined by*

$$w(x, y, t, z) = 3uv \tag{2.12}$$

solves the (3+1)-dimensional nonlinear evolution equation (1.3).

Proof. Using (2.1)–(2.3), a direct calculation arrives at

$$\begin{aligned}
&w_y = 3(uv_{xx} - u_{xx}v) \qquad \partial_x^{-1} w_y = 3(uv_x - u_x v)\\
&w_{xxxy} = 3(uv_{xxxx} - u_{xxxx}v)_x + 6(u_x v_{xxx} - u_{xxx}v_x)_x\\
&w_t = 3(u_{xxx}v + uv_{xxx}) - 2ww_x \qquad w_x \partial_x^{-1} w_y = 9\left(u^2 v_x^2 - u_x^2 v^2\right)\\
&w_{ty} = 3(uv_{xxxx} - u_{xxxx}v)_x + 3(u_{xxx}v_x - u_x v_{xxx})_x - 2\left(w\partial_x^{-1} w_y\right)_{xx}\\
&w_{zx} = 3(uv_{xxxx} - u_{xxxx}v)_x - 2\left(ww_y + w_x \partial_x^{-1} w_y\right)_x
\end{aligned} \tag{2.13}$$

which give the (3+1)-dimensional evolution equation (1.3). □

3. Decomposition of the AKNS hierarchy

In what follows, we shall construct the AKNS hierarchy, which plays the role of a bridge in the process of reducing the (3+1)-dimensional evolution equation (1.1) and the (2+1)-dimensional coupled KP equations (1.2) to solvable ordinary differential equations . Let us consider the Lenard gradient sequence s_j, $-1 \leqslant j \in \mathbb{Z}$ by the recursion relation

$$\begin{aligned}
&s_{l+1}^{(1)} = \partial_x s_l^{(1)} - 2vs_l^{(3)}\\
&s_{l+1}^{(2)} = -\partial_x s_l^{(2)} - 2us_l^{(3)}\\
&s_{l+1}^{(3)} = \sum_{j=0}^{l} \left(s_j^{(1)} s_{l-j}^{(2)} + s_j^{(3)} s_{l-j}^{(3)}\right) \qquad l \geqslant -1
\end{aligned} \tag{3.1}$$

with $s_{-1} = \left(0, 0, -\frac{1}{2}\right)^T$. It is easy to see that s_j is uniquely determined by (3.1) and the first few members are

$$s_0 = \begin{pmatrix} v \\ u \\ 0 \end{pmatrix} \qquad s_1 = \begin{pmatrix} v_x \\ -u_x \\ uv \end{pmatrix}$$

$$s_2 = \begin{pmatrix} v_{xx} - 2uv^2 \\ u_{xx} - 2u^2 v \\ uv_x - u_x v \end{pmatrix} \qquad s_3 = \begin{pmatrix} v_{xxx} - 6uvv_x \\ -u_{xxx} + 6uvu_x \\ u_{xx}v + uv_{xx} - u_x v_x - 3u^2 v^2 \end{pmatrix}$$

$$s_4 = \begin{pmatrix} v_{xxxx} - 8uvv_{xx} - 6uv_x^2 - 4vu_x v_x - 2v^2 u_{xx} + 6u^2 v^3 \\ u_{xxxx} - 8uvu_{xx} - 6u_x^2 v - 4uu_x v_x - 2u^2 v_{xx} + 6u^3 v^2 \\ uv_{xxx} - u_{xxx}v + u_{xx}v_x - u_x v_{xx} + 6uu_x v^2 - 6u^2 vv_x \end{pmatrix}.$$

From (3.1), we have

$$\partial_x s^{(3)}_{l+1} = \sum_{j=0}^{l} \left(s^{(1)}_{jx} s^{(2)}_{l-j} + s^{(1)}_{j} s^{(2)}_{l-j,x} + \partial_x \left(s^{(3)}_{j} s^{(3)}_{l-j} \right) \right)$$
$$= \sum_{j=0}^{l} \left(s^{(1)}_{j+1} s^{(2)}_{l-j} - s^{(1)}_{j} s^{(2)}_{l+1-j} + 2v s^{(3)}_{j} s^{(2)}_{l-j} - 2u s^{(1)}_{j} s^{(3)}_{l-j} + \partial_x \left(s^{(3)}_{j} s^{(3)}_{l-j} \right) \right)$$
$$= u s^{(1)}_{l+1} - v s^{(2)}_{l+1} + 2 \sum_{j=0}^{l} s^{(3)}_{l-j} \left(v s^{(2)}_{j} - u s^{(1)}_{j} + \partial_x s^{(3)}_{j} \right) \tag{3.2}$$

which implies by induction that

$$-u s^{(1)}_{l} + v s^{(2)}_{l} + \partial_x s^{(3)}_{l} = 0 \qquad l \geqslant -1. \tag{3.3}$$

Equations (3.1) and (3.3) can be written as the Lenard equation

$$K s_{l-1} = J s_l \qquad J s_{-1} = 0 \qquad l \geqslant 0 \tag{3.4}$$

with two operators

$$K = \begin{pmatrix} 0 & \partial_x & 2u \\ \partial_x & 0 & -2v \\ -u & v & \partial_x \end{pmatrix} \qquad J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -u & v & \partial_x \end{pmatrix}.$$

Consider the AKNS the spectral problem and the auxiliary problem

$$\varphi_x = U \varphi \qquad U = \begin{pmatrix} -\frac{1}{2}\lambda & u \\ v & \frac{1}{2}\lambda \end{pmatrix} \tag{3.5}$$

$$\varphi_{t_m} = V^{(m)} \varphi \qquad V^{(m)} = \begin{pmatrix} V^{(m)}_{11} & V^{(m)}_{12} \\ V^{(m)}_{21} & -V^{(m)}_{11} \end{pmatrix} \tag{3.6}$$

where

$$V^{(m)}_{11} = \sum_{j=0}^{m} s^{(3)}_{j-1} \lambda^{m-j} \qquad V^{(m)}_{12} = \sum_{j=1}^{m} s^{(2)}_{j-1} \lambda^{m-j} \qquad V^{(m)}_{21} = \sum_{j=1}^{m} s^{(1)}_{j-1} \lambda^{m-j}.$$

Then the compatibility condition of (3.5) and (3.6) is the Lax equation, $U_{t_m} - V^{(m)}_x + [U, V^{(m)}] = 0$, which is equivalent to the AKNS hierarchy

$$\left(u_{t_m}, v_{t_m} \right)^T = X_m \qquad m \geqslant 0 \tag{3.7}$$

where the AKNS vector field

$$X_j = \sigma \hat{s}_j \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \hat{s}_j = \begin{pmatrix} s^{(1)}_j \\ s^{(2)}_j \end{pmatrix}.$$

The first three nontrivial members in the hierarchy (3.7) are exactly equations (2.1)–(2.3) with $t_2 = y, t_3 = t, t_4 = z$.

Assume that (3.5) and (3.6) have two basic solutions $\psi = (\psi_1, \psi_2)^T$ and $\phi = (\phi_1, \phi_2)^T$. We introduce a Lax matrix W of three functions f, g, h by

$$W = \frac{1}{2}(\phi \psi^T + \psi \phi^T)\sigma = \begin{pmatrix} f & g \\ h & -f \end{pmatrix} \tag{3.8}$$

which satisfies the Lax equations

$$W_x = [U, W] \qquad W_{t_m} = [V^{(m)}, W]. \tag{3.9}$$

This means that the function det W is a constant independent of x and t_m. Equations (3.10) can be written as

$$f_x = uh - vg \qquad g_x = -\lambda g - 2uf \qquad h_x = 2vf + \lambda h \tag{3.10}$$

and

$$f_{t_m} = hV_{12}^{(m)} - gV_{21}^{(m)} \qquad g_{t_m} = 2gV_{11}^{(m)} - 2fV_{12}^{(m)} \qquad h_{t_m} = 2fV_{21}^{(m)} - 2hV_{11}^{(m)}. \tag{3.11}$$

Now suppose that the functions f, g and h are finite-order polynomials in λ:

$$f = \sum_{j=0}^{N+1} f_{j-1}\lambda^{N+1-j} \qquad g = \sum_{j=0}^{N+1} g_{j-1}\lambda^{N+1-j} \qquad h = \sum_{j=0}^{N+1} h_{j-1}\lambda^{N+1-j}. \tag{3.12}$$

From (3.10) and (3.12), we have

$$KG_{j-1} = JG_j \qquad JG_{-1} = 0 \tag{3.13}$$

$$KG_N = 0 \qquad G_j = (h_j, g_j, f_j)^T. \tag{3.14}$$

Equations (3.13) and (3.14) imply

$$-uh_j + vg_j + \partial_x f_j = 0 \qquad JG_j = (*, *, 0)^T. \tag{3.15}$$

Note that the equation $JG_{-1} = 0$ has the general solution

$$G_{-1} = \alpha_0 s_{-1} \tag{3.16}$$

with a constant of integration α_0, which shows ker$J = \{as_{-1}|\forall a\}$. Acting with the operator $(J^{-1}K)^{k+1}$ upon G_{-1} in (3.14), we obtain from (3.13) and (3.4) that

$$G_k = \sum_{j=0}^{k+1} \alpha_j s_{k-j} \qquad -1 \leqslant k \leqslant N \tag{3.17}$$

where $\alpha_1, \ldots, \alpha_{k+1}$ are constants of integration. Substituting (3.17) into (3.14) yields the stationary AKNS equation

$$\alpha_0 X_{N+1} + \alpha_1 X_N + \cdots + \alpha_{N+1} X_0 = 0. \tag{3.18}$$

4. Solvable ordinary differential equations

In this section, we shall decompose the AKNS hierarchy into solvable ordinary differential equations. Without any loss of generality we can set $\alpha_0 = 1$. Then from (3.17), we have

$$\begin{aligned} &f_{-1} = -\tfrac{1}{2} \qquad f_0 = -\tfrac{1}{2}\alpha_1 \qquad f_1 = s_1^{(3)} - \tfrac{1}{2}\alpha_2 \\ &g_{-1} = 0 \qquad g_0 = s_0^{(2)} \qquad h_{-1} = 0 \qquad h_0 = s_0^{(1)} \end{aligned} \tag{4.1}$$

$$\begin{aligned} f_k &= s_k^{(3)} + \alpha_1 s_{k-1}^{(3)} + \cdots + \alpha_{k-1} s_1^{(3)} - \tfrac{1}{2}\alpha_{k+1} \\ g_k &= s_k^{(2)} + \alpha_1 s_{k-1}^{(2)} + \cdots + \alpha_k s_0^{(2)} \\ h_k &= s_k^{(1)} + \alpha_1 s_{k-1}^{(1)} + \cdots + \alpha_k s_0^{(1)} \qquad k \geqslant 1. \end{aligned} \tag{4.2}$$

Equations (4.1) and (4.2) can be written as

$$\begin{aligned} s_k^{(1)} &= \gamma_0 h_k + \gamma_1 h_{k-1} + \cdots + \gamma_k h_0 \\ s_k^{(2)} &= \gamma_0 g_k + \gamma_1 g_{k-1} + \cdots + \gamma_k g_0 \qquad k \geqslant 1 \\ s_k^{(3)} &= \gamma_0 f_k + \gamma_1 f_{k-1} + \cdots + \gamma_{k-1} f_1 + \pi_k \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} &\gamma_0 = 1 \qquad \gamma_1 = -\alpha_1 \qquad \gamma_2 = -\alpha_1\gamma_1 - \alpha_2, \ldots \\ &\gamma_k = -\alpha_1\gamma_{k-1} - \alpha_2\gamma_{k-2} - \cdots - \alpha_{k-1}\gamma_1 - \alpha_k \\ &\pi_1 = \tfrac{1}{2}\alpha_2 \qquad \pi_2 = \tfrac{1}{2}\alpha_3 - \alpha_1\pi_1, \ldots \\ &\pi_{k+1} = \tfrac{1}{2}\alpha_{k+2} - \alpha_1\pi_k - \alpha_2\pi_{k-1} - \cdots - \alpha_k\pi_1. \end{aligned} \tag{4.4}$$

Now we write g and h as finite products, which take the forms

$$g = u\prod_{i=1}^{N}(\lambda - \mu_i) \qquad h = v\prod_{i=1}^{N}(\lambda - \nu_i) \tag{4.5}$$

to define the elliptic coordinates $\{\mu_i\}$ and $\{\nu_i\}$. Noting (4.1), (4.2) and (3.12), we get by comparing the coefficients of the same power for λ that

$$\begin{aligned} g_1 &= -u\sum_{j=1}^{N}\mu_j \qquad h_1 = -v\sum_{j=1}^{N}\nu_j \\ g_2 &= u\sum_{i<j}\mu_i\mu_j \qquad h_2 = v\sum_{i<j}\nu_i\nu_j \end{aligned} \tag{4.6}$$

$$\begin{aligned} g_l &= (-1)^l u \sum_{j_1<j_2<\cdots<j_l}\mu_{j_1}\mu_{j_2}\cdots\mu_{j_l} \\ h_l &= (-1)^l v \sum_{j_1<j_2<\cdots<j_l}\nu_{j_1}\nu_{j_2}\cdots\nu_{j_l} \qquad 1 \leqslant l \leqslant N. \end{aligned} \tag{4.7}$$

By using (4.6) and (4.2), we arrive at

$$\partial_x \ln u = \alpha_1 + \sum_{j=1}^{N}\mu_j \qquad \partial_x \ln v = -\alpha_1 - \sum_{j=1}^{N}\nu_j \tag{4.8}$$

$$\begin{aligned} 2uv &= \frac{1}{2}\left(\sum_{j=1}^{N}\mu_j\right)^2 + \frac{1}{2}\sum_{j=1}^{N}\mu_j^2 + \partial_x\sum_{j=1}^{N}\mu_j + \alpha_1\sum_{j=1}^{N}\mu_j + \alpha_2 \\ 2uv &= \frac{1}{2}\left(\sum_{j=1}^{N}\nu_j\right)^2 + \frac{1}{2}\sum_{j=1}^{N}\nu_j^2 - \partial_x\sum_{j=1}^{N}\nu_j + \alpha_1\sum_{j=1}^{N}\nu_j + \alpha_2 \end{aligned} \tag{4.9}$$

with the help of equality

$$\begin{aligned} 2\sum_{i<j}\xi_i\xi_j &= \left(\sum_{j=1}^{N}\xi_j\right)^2 - \sum_{j=1}^{N}\xi_j^2 \\ 6\sum_{i<j<k}\xi_i\xi_j\xi_k &= \left(\sum_{j=1}^{N}\xi_j\right)^3 + 2\sum_{j=1}^{N}\xi_j^3 - 3\sum_{j=1}^{N}\xi_j\left(\sum_{j=1}^{N}\xi_j^2\right). \end{aligned} \tag{4.10}$$

From (4.3), equation (3.7) can be written as

$$\begin{aligned} u_{t_m} &= -\gamma_0 g_m - \gamma_1 g_{m-1} - \cdots - \gamma_m g_0 \\ v_{t_m} &= \gamma_0 h_m + \gamma_1 h_{m-1} + \cdots + \gamma_m h_0 \end{aligned} \tag{4.11}$$

where g_l and h_l are given by (4.6) and (4.7). Note that the function $\det W$ is a $(2N+2)$th-order polynomial in λ, whose coefficients are constants of the x-flow and t_m-flow. We have

$$-\det \mathrm{W} = f^2 + gh = \frac{1}{4}\prod_{j=1}^{2N+2}(\lambda - \lambda_j) = \frac{1}{4}R(\lambda). \tag{4.12}$$

Substituting (3.12) into the above expression and comparing the coefficients of λ^{2N+1}, $\lambda^{2N}, \ldots, \lambda^{N+1}$, we obtain

$$2f_{-1}f_0 = -\frac{1}{4}\sum_{j=1}^{2N+2}\lambda_j \qquad \alpha_1 = -\frac{1}{2}\sum_{j=1}^{2N+2}\lambda_j \tag{4.13}$$

$$2f_{-1}f_l + \sum_{j=0}^{l-1}(f_j f_{l-j-1} + g_j h_{l-j-1}) = \frac{1}{4}(-1)^{l+1}\sum_{j_1<\cdots<j_{l+1}}\lambda_{j_1}\cdots\lambda_{j_{l+1}} \qquad 1 \leqslant l \leqslant N. \tag{4.14}$$

Using (4.2), it is easy to calculate that

$$\sum_{j=0}^{l-1} g_j h_{l-j-1} = \sum_{n=0}^{l-1}\sum_{k=0}^{l-1-n}\alpha_k\alpha_{l-n-1-k}\sum_{i=0}^{n}s_i^{(1)}s_{n-i}^{(2)} \qquad (\alpha_0 = 1) \tag{4.15}$$

$$\sum_{j=0}^{l-1} f_j f_{l-j-1} = \sum_{n=0}^{l-1}\sum_{k=0}^{l-1-n}\alpha_k\alpha_{l-n-1-k}\sum_{i=0}^{n}s_i^{(3)}s_{n-i}^{(3)} - \sum_{j=0}^{l-1}\sum_{k=0}^{j}\alpha_{l-j}\alpha_{j-k}s_k^{(3)} + \frac{1}{4}\sum_{j=0}^{l-1}\alpha_{j+1}\alpha_{l-j} \tag{4.16}$$

$$\sum_{j=0}^{l-1}\sum_{k=0}^{j}\alpha_{l-j}\alpha_{j-k}s_k^{(3)} = \sum_{n=0}^{l-1}\sum_{k=0}^{l-1-n}\alpha_k\alpha_{l-n-1-k}s_{n+1}^{(3)} + \sum_{k=0}^{l}\alpha_k\alpha_{l-k}s_0^{(3)} - \alpha_0\sum_{k=0}^{l}\alpha_{l-k}s_k^{(3)}. \tag{4.17}$$

By utilizing (4.15)–(4.17), (3.1), (4.2) and (4.14), we have

$$\alpha_{l+1} + \frac{1}{2}\sum_{j=1}^{l}\alpha_j\alpha_{l+1-j} = (-1)^{l+1}\frac{1}{2}\sum_{j_1<\cdots<j_{l+1}}\lambda_{j_1}\cdots\lambda_{j_{l+1}} \qquad 1 \leqslant l \leqslant N \tag{4.18}$$

from which α_j $(1 \leqslant l \leqslant N)$ can be explicitly represented by the constants $\lambda_1, \ldots, \lambda_{2N+2}$. Noting (4.5) and (3.10), we get

$$\begin{aligned} g_x|_{\lambda=\mu_k} &= -u\mu_{kx}\prod_{i=1,i\neq k}^{N}(\mu_k - \mu_i) = -2uf|_{\lambda=\mu_k} \\ h_x|_{\lambda=\nu_k} &= -v\nu_{kx}\prod_{i=1,i\neq k}^{N}(\nu_k - \nu_i) = 2vf|_{\lambda=\nu_k} \qquad 1 \leqslant k \leqslant N \end{aligned} \tag{4.19}$$

which, together with (4.12), imply the evolution of the elliptic coordinates along the x-flow:

$$\begin{aligned} \frac{\mu_{kx}}{\sqrt{R(\mu_k)}} &= \frac{1}{\prod_{i=1,i\neq k}^{N}(\mu_k - \mu_i)} \\ \frac{\nu_{kx}}{\sqrt{R(\nu_k)}} &= -\frac{1}{\prod_{i=1,i\neq k}^{N}(\nu_k - \nu_i)} \end{aligned} \qquad 1 \leqslant k \leqslant N. \tag{4.20}$$

We obtain from (3.6) and (4.3) that

$$\begin{aligned} V_{12}^{(m)}\big|_{\lambda=\mu_k} &= \sum_{n=1}^{m}\sum_{l=0}^{n-1}\gamma_{n-l-1}g_l\mu_k^{m-n} \\ V_{21}^{(m)}\big|_{\lambda=\nu_k} &= \sum_{n=1}^{m}\sum_{l=0}^{n-1}\gamma_{n-l-1}h_l\nu_k^{m-n}. \end{aligned} \tag{4.21}$$

In a way similar to the calculation of (4.20), we arrive at the evolution of $\{\mu_k\}$ and $\{\nu_k\}$ along the t_m-flow:

$$\begin{aligned} \frac{\mu_{kt_m}}{\sqrt{R(\mu_k)}} &= \frac{\sum_{n=1}^{m}\sum_{l=0}^{n-1}\gamma_{n-l-1}u^{-1}g_l\mu_k^{m-n}}{\prod_{i=1,i\neq k}^{N}(\mu_k-\mu_i)} \\ \frac{\nu_{kt_m}}{\sqrt{R(\nu_k)}} &= \frac{\sum_{n=1}^{m}\sum_{l=0}^{n-1}\gamma_{n-l-1}v^{-1}h_l\nu_k^{m-n}}{\prod_{i=1,i\neq k}^{N}(\nu_k-\nu_i)} \end{aligned} \qquad 1\leqslant k\leqslant N \quad 2\leqslant m\leqslant N \tag{4.22}$$

where g_l and h_l are given by (4.6) and (4.7).

Therefore, if the $2N+2$ distinct parameters $\lambda_1,\ldots,\lambda_{2N+2}$ are given, and let μ_k and ν_k be distinct solutions of ordinary differential equations (4.20) and (4.22), then (u, v) determined by (4.8) and (4.11) is a solution of the AKNS equation (3.7). This means that the function $(\hat{q},\hat{r})$ by (2.4), the function (p,q,r) by (2.10) $(2\leqslant m\leqslant 3)$ and the function w by (2.12) $(2\leqslant m\leqslant 4)$ are solutions of the (2+1)-dimensional breaking soliton equation (1.1), the coupled KP equation (1.2) and the (3+1)-dimensional evolution equation (1.3), respectively.

5. Algebraic-geometrical solutions

Let us consider the Riemann surface Γ of the hyperelliptic curve $\zeta^2 = R(\lambda)$, $R(\lambda) = \prod_{j=1}^{2N+2}(\lambda-\lambda_j)$, of genus N. On Γ there are two infinite points ∞_1 and ∞_2, which are not branch points of Γ. Equip Γ with the canonical basis of cycles: $a_1,\ldots,a_N; b_1,\ldots,b_N$, and the holomorphic differentials

$$\tilde{\omega}_l = \frac{\lambda^{l-1}d\lambda}{\sqrt{R(\lambda)}} \qquad 1\leqslant l\leqslant N.$$

Then the period matrices A and B defined by

$$A_{ij} = \int_{a_j}\tilde{\omega}_i \qquad B_{ij} = \int_{b_j}\tilde{\omega}_i$$

are invertible [32, 33]. Let $C = A^{-1}$, $\tau = A^{-1}B$. The matrix τ is symmetric $(\tau_{ij} = \tau_{ji})$ and has positive definite imaginary part $(\operatorname{Im}\tau > 0)$. Then the Riemann theta function of Γ is defined as

$$\theta(\xi|\tau) = \sum_{z\in\mathbb{Z}^N}\exp(\pi\sqrt{-1}\langle\tau z, z\rangle + 2\pi\sqrt{-1}\langle\xi, z\rangle) \qquad \xi = (\xi_1,\ldots,\xi)^T \in\mathbb{C}^N$$

where $\langle .,.\rangle$ represents the inner-product, $\langle\xi,\varsigma\rangle = \sum_{j=1}^{N}\xi_j\varsigma_j$. If we normalize $\tilde{\omega}_l$ into the new basis ω_j

$$\omega_j = \sum_{l=1}^{N}C_{jl}\tilde{\omega}_l \qquad 1\leqslant j\leqslant N$$

then we have

$$\int_{a_i}\omega_j = \delta_{ji} \qquad \int_{b_i}\omega_j = \tau_{ji}.$$

Now we introduce the Abel map $\mathcal{A}(p)$

$$\mathcal{A}(p) = \int_{p_0}^{p} \omega \qquad \mathcal{A}\left(\sum n_k p_k\right) = \sum n_k \mathcal{A}(p_k)$$

and the Abel–Jacobi coordinates

$$\rho^{(1)} = \mathcal{A}\left(\sum_{k=1}^{N} p(\mu_k)\right) = \sum_{k=1}^{N} \int_{p_0}^{p(\mu_k)} \omega \tag{5.1}$$

$$\rho^{(2)} = \mathcal{A}\left(\sum_{k=1}^{N} p(\nu_k)\right) = \sum_{k=1}^{N} \int_{p_0}^{p(\nu_k)} \omega \tag{5.2}$$

where

$$p(\mu_k) = (\lambda = \mu_k, \zeta = \sqrt{R(\mu_k)}) \qquad p(\nu_k) = (\lambda = \nu_k, \zeta = \sqrt{R(\nu_k)}) \in \Gamma$$

and p_0 is chosen as a base point on Γ. The components of the Abel–Jacobi coordinates in (5.1) and (5.2) read

$$\rho_j^{(1)} = \sum_{k=1}^{N} \int_{p_0}^{p(\mu_k)} \omega_j = \sum_{k=1}^{N} \sum_{l=1}^{N} C_{jl} \int_{\lambda(p_0)}^{\mu_k} \frac{\lambda^{l-1}\,\mathrm{d}\lambda}{\sqrt{R(\lambda)}} \qquad 1 \leqslant j \leqslant N \tag{5.3}$$

$$\rho_j^{(2)} = \sum_{k=1}^{N} \int_{p_0}^{p(\nu_k)} \omega_j = \sum_{k=1}^{N} \sum_{l=1}^{N} C_{jl} \int_{\lambda(p_0)}^{\nu_k} \frac{\lambda^{l-1}\,\mathrm{d}\lambda}{\sqrt{R(\lambda)}} \qquad 1 \leqslant j \leqslant N \tag{5.4}$$

where $\lambda(p_0)$ is the local coordinates of p_0. From the first expression of (4.19), we get

$$\partial_x \rho_j^{(1)} = \sum_{l=1}^{N} \sum_{k=1}^{N} C_{jl} \frac{\mu_k^{l-1}\mu_{kx}}{\sqrt{R(\mu_k)}} = \sum_{l=1}^{N} \sum_{k=1}^{N} \frac{\mu_k^{l-1} C_{jl}}{\prod_{i\neq k}^{N}(\mu_k - \mu_i)}$$

which implies

$$\partial_x \rho_j^{(1)} = C_{jN} = \Omega_j^{(0)} \qquad 1 \leqslant j \leqslant N \tag{5.5}$$

with the help of the following equality:

$$\sum_{k=1}^{N} \frac{\mu_k^{l-1}}{\prod_{i\neq k}^{N}(\mu_k - \mu_i)} = \delta_{lN} \qquad 1 \leqslant l \leqslant N. \tag{5.6}$$

In a similar way to the calculation of [21], we obtain from (5.3), (5.4) and (4.21) that

$$\partial_{t_m} \rho_j^{(1)} = \Omega_j^{(m-1)} \qquad \partial_{t_m} \rho_j^{(2)} = -\Omega_j^{(m-1)} \qquad 1 \leqslant j \leqslant N \quad 2 \leqslant m \leqslant 4 \tag{5.7}$$

with

$$\Omega_j^{(m-1)} = C_{j,N-m+1} + \gamma_1 C_{j,N-m+2} + \cdots + \gamma_{m-1} C_{jN}. \tag{5.8}$$

On the basis of these results, it is easy to see that $\rho_j^{(1)}$ and $\rho_j^{(2)}$ are linear functions:

$$\rho_j^{(1)} = \Omega_j^{(0)} x + \sum_{k=2}^{m} \Omega_j^{(k-1)} t_k + \gamma_j^{(1)} \qquad 1 \leqslant j \leqslant N \quad 2 \leqslant m \leqslant 4 \tag{5.9}$$

$$\rho_j^{(2)} = -\Omega_j^{(0)} x - \sum_{k=2}^{m} \Omega_j^{(k-1)} t_k - \gamma_j^{(2)} \qquad 1 \leqslant j \leqslant N \quad 2 \leqslant m \leqslant 4 \tag{5.10}$$

where $\gamma_j^{(i)}$ $(i = 1, 2)$ are constants,

$$\gamma_j^{(1)} = \sum_{k=1}^{N} \int_{p_0}^{p(\mu_k(0))} \omega_j \qquad \gamma_j^{(2)} = -\sum_{k=1}^{N} \int_{p_0}^{p(\nu_k(0))} \omega_j.$$

Especially, for the (2+1)-dimensional breaking soliton equation (1.1) and the coupled KP equation (1.2) we have

$$\begin{aligned} \rho_j^{(1)} &= \Omega_j^{(0)} x + \Omega_j^{(1)} y + \Omega_j^{(2)} t + \gamma_j^{(1)} \\ \rho_j^{(2)} &= -\Omega_j^{(0)} x - \Omega_j^{(1)} y - \Omega_j^{(2)} t - \gamma_j^{(2)} \qquad 1 \leqslant j \leqslant N. \end{aligned} \tag{5.11}$$

Moreover, for the (3+1)-dimensional nonlinear evolution equation (1.3), we have

$$\begin{aligned} \rho_j^{(1)} &= \Omega_j^{(0)} x + \Omega_j^{(1)} y + \Omega_j^{(2)} t + \Omega_j^{(3)} z + \gamma_j^{(1)} \\ \rho_j^{(2)} &= -\Omega_j^{(0)} x - \Omega_j^{(1)} y - \Omega_j^{(2)} t - \Omega_j^{(3)} z - \gamma_j^{(2)} \qquad 1 \leqslant j \leqslant N. \end{aligned} \tag{5.12}$$

According to the Riemann theorem [32, 33], there exists a constant vector $M^{(l)} \in \mathbb{C}^N$ such that the function

$$F^{(l)}(\lambda) = \theta(\mathcal{A}(p(\lambda)) - \rho^{(l)} - M^{(l)}) \qquad l = 1, 2$$

has exactly N zeros at $\mu_1, \ldots, \mu_N$ for $l = 1$ or $\nu_1, \ldots, \nu_N$ for $l = 2$, and we have the inversion formula:

$$\begin{aligned} \sum_{j=1}^N \mu_j^k &= I_k(\Gamma) - \sum_{s=1}^2 \operatorname*{Res}_{\lambda=\infty_s} \lambda^k \, \mathrm{d} \ln F^{(1)}(\lambda) \\ \sum_{j=1}^N \nu_j^k &= I_k(\Gamma) - \sum_{s=1}^2 \operatorname*{Res}_{\lambda=\infty_s} \lambda^k \, \mathrm{d} \ln F^{(2)}(\lambda) \end{aligned} \tag{5.13}$$

where I_k is the constant independent of $\rho^{(l)}$

$$I_k(\Gamma) = \sum_{j=1}^N \int_{a_j} \lambda^k \omega_j.$$

In what follows, we shall compute the residues in (5.13) for $k = 1, \ldots, 4$. Through tedious calculations, we obtain

$$\begin{aligned} &\operatorname*{Res}_{\lambda=\infty_s} \lambda \, \mathrm{d} \ln F^{(l)}(\lambda) = (-1)^{s-1} \sum_{j=1}^N \Omega_j^{(0)} D_j \ln \theta_s^{(l)} \qquad 1 \leqslant l \leqslant 2 \quad 1 \leqslant s \leqslant 2 \\ &\operatorname*{Res}_{\lambda=\infty_s} \lambda^2 \, \mathrm{d} \ln F^{(l)}(\lambda) = (-1)^{s-1} \sum_{j=1}^N \Omega_j^{(1)} D_j \ln \theta_s^{(l)} + \sum_{j=1}^N \sum_{k=1}^N \Omega_j^{(0)} \Omega_k^{(0)} D_{jk} \ln \theta_s^{(l)} \\ &\operatorname*{Res}_{\lambda=\infty_s} \lambda^3 \, \mathrm{d} \ln F^{(l)}(\lambda) = (-1)^{s-1} \sum_{j=1}^N \Omega_j^{(2)} D_j \ln \theta_s^{(l)} + \frac{3}{2} \sum_{j=1}^N \sum_{k=1}^N \Omega_j^{(0)} \Omega_k^{(1)} D_{jk} \ln \theta_s^{(l)} \\ &\qquad + \frac{1}{2} (-1)^{s-1} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \Omega_i^{(0)} \Omega_j^{(0)} \Omega_k^{(0)} D_{ijk} \ln \theta_s^{(l)} \end{aligned} \tag{5.14}$$

$$\begin{aligned} \operatorname*{Res}_{\lambda=\infty_s} \lambda^4 \, \mathrm{d} \ln F^{(l)}(\lambda) &= (-1)^{s-1} \sum_{j=1}^N \Omega_k^{(3)} D_{jk} \ln \theta_s^{(l)} + \frac{4}{3} \sum_{j=1}^N \sum_{k=1}^N \Omega_j^{(0)} \Omega_k^{(2)} D_{jk} \ln \theta_s^{(l)} \\ &\quad + \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N \Omega_j^{(1)} \Omega_k^{(1)} D_{jk} \ln \theta_s^{(l)} + (-1)^{s-1} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \Omega_i^{(0)} \Omega_j^{(0)} \Omega_k^{(1)} D_{ijk} \ln \theta_s^{(l)} \\ &\quad + \frac{1}{6} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{n=1}^N \Omega_i^{(0)} \Omega_j^{(0)} \Omega_k^{(0)} \Omega_n^{(0)} D_{ijkn} \ln \theta_s^{(l)} \end{aligned} \tag{5.15}$$

with

$$\theta_s^{(1)} = \theta\left(\Omega^{(0)}x + \sum_{k=2}^{m}\Omega^{(k-1)}t_k + \Upsilon^{(s)}\right) \qquad \theta_s^{(2)} = \theta\left(\Omega^{(0)}x + \sum_{k=2}^{m}\Omega^{(k-1)}t_k + \Lambda^{(s)}\right)$$

where D_j signifies a derivative with respect to the jth argument of the theta function, $D_{ij} = D_i D_j$, $D_{ijk} = D_i D_j D_k$, $D_{ijkn} = D_i D_j D_k D_n$, and

$$\Omega^{(i)} = \left(\Omega_1^{(i)}, \ldots, \Omega_N^{(i)}\right)^T \qquad \Upsilon^{(s)} = \left(\Upsilon_1^{(s)}, \ldots, \Upsilon_N^{(s)}\right)^T \qquad \Lambda^{(s)} = \left(\Lambda_1^{(s)}, \ldots, \Lambda_N^{(s)}\right)^T$$

$$\Upsilon_j^{(s)} = \gamma_j^{(1)} + M_j^{(1)} + \int_{\infty_s}^{p_0}\omega_j \qquad \Lambda_j^{(s)} = \gamma_j^{(2)} - M_j^{(2)} - \int_{\infty_s}^{p_0}\omega_j$$

$$0 \leqslant i \leqslant m-1 \quad 1 \leqslant j \leqslant N.$$

Here we use the property that the theta function is an even one. Equations (5.13)–(5.15) imply the equalities

$$\sum_{j=1}^{N}\mu_j = I_1(\Gamma) + \partial_x \ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}} \qquad \sum_{j=1}^{N}\nu_j = I_1(\Gamma) + \partial_x \ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}} \tag{5.16}$$

$$\begin{aligned}\sum_{j=1}^{N}\mu_j^2 &= I_2(\Gamma) + \partial_{t_2}\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}} - \partial_x^2 \ln\theta_1^{(1)}\theta_2^{(1)}\\ \sum_{j=1}^{N}\nu_j^2 &= I_2(\Gamma) + \partial_{t_2}\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}} - \partial_x^2 \ln\theta_1^{(2)}\theta_2^{(2)}\end{aligned} \tag{5.17}$$

$$\begin{aligned}\sum_{j=1}^{N}\mu_j^3 &= I_3(\Gamma) + \left(\partial_{t_3} + \frac{1}{2}\partial_x^3\right)\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}} - \frac{3}{2}\partial_x\partial_{t_2}\ln\theta_1^{(1)}\theta_2^{(1)}\\ \sum_{j=1}^{N}\nu_j^3 &= I_3(\Gamma) + \left(\partial_{t_3} + \frac{1}{2}\partial_x^3\right)\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}} - \frac{3}{2}\partial_x\partial_{t_2}\ln\theta_1^{(2)}\theta_2^{(2)}.\end{aligned} \tag{5.18}$$

Using (4.6), (4.7), (5.16)–(5.18), and noting (4.10), the first expressions of (4.8) and (4.11) can be written as

$$\partial_x \ln u = N_1 + \partial_x \ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}} \qquad \partial_{t_m}\ln u = N_m + \partial_{t_m}\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}} \tag{5.19}$$

where N_1, N_2 and N_3 are given by

$$\begin{aligned}N_1 &= I_1 - \gamma_1\\ N_2 &= -\frac{1}{2}\left(\partial_{t_2} + \partial_x^2\right)\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}} - \partial_x^2\ln\theta_1^{(1)} - \frac{1}{2}\left(I_1 + \partial_x\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}}\right)^2 + \gamma_1\left(I_1 + \partial_x\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}}\right) + \frac{1}{2}I_2 - \gamma_2\\ N_3 &= -\frac{2}{3}\left(\partial_{t_3} - \frac{1}{4}\partial_x^3 + \frac{3}{4}\partial_x\partial_{t_2}\right)\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}} - \partial_x\partial_{t_2}\ln\theta_1^{(1)} + \gamma_2\left(I_1 + \partial_x\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}}\right)\\ &\quad - \frac{1}{2}\left(N_1 + \partial_x\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}}\right)\left[I_2 + \left(\partial_{t_2} - \partial_x^2\right)\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}} - 2\partial_x^2\ln\theta_1^{(1)}\right]\\ &\quad + \frac{1}{6}\left(I_1 + \partial_x\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}}\right)^2\left(I_1 - 3\gamma_1 + \partial_x\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}}\right) + \frac{1}{3}I_3 - \gamma_3.\end{aligned} \tag{5.20}$$

Proposition 5.1.

$$\tfrac{1}{2}N_2 = -uv - \partial_x^2 \ln\theta_1^{(1)} + \tfrac{1}{2}\big(\gamma_1^2 - 2\gamma_2 + I_2\big) \tag{5.21}$$

$$\tfrac{1}{3}N_3 = u_x v - uv_x - \partial_x\partial_{t_2}\ln\theta_1^{(1)} + \gamma_1\gamma_2 - \gamma_3 + \tfrac{1}{3}\big(I_3 - \gamma_1^3\big). \tag{5.22}$$

Noting (5.19)–(5.22), it is easy to see that $N_{mx} = 0$, $N_{1t_m} = 0$, $N_{2t_2} = 0$. Through lengthy calculations, we arrive at the following equalities:

$$\begin{aligned}\big(8\partial_x\partial_{t_3} + 3\partial_{t_2}^2\big)\ln\theta_1^{(1)} &= -11(u_{xx}v + uv_{xx}) + 14u_x v_x + 30u^2v^2 - 3N_4\\ &\quad + 6\gamma_2^2 + 12\gamma_1\gamma_3 - 12\gamma_1^2\gamma_2 + 12\gamma_4 + 3\gamma_1^4 + 3I_4\end{aligned} \tag{5.23}$$

$$\begin{aligned}\big(3\partial_x\partial_{t_4} + 2\partial_{t_2}\partial_{t_3}\big)\ln\theta_1^{(1)} &= 5(u_{xxx}v - uv_{xxx}) - 7(u_{xx}v_x - u_x v_{xx})\\ &\quad - 30(uu_x v^2 - u^2vv_x) - \tfrac{6}{5}N_5 + \text{const}.\end{aligned} \tag{5.24}$$

Using (5.21)–(5.24), we have that $N_{2t_3} = 0$, $N_{3t_2} = 0$. Similarly, through tedious calculations we can prove that $N_{2t_4} = 0$, $N_{3t_3} = 0$.

Proposition 5.2. *N_1 and N_2 are constants independent of x, t_2, t_3, t_4. And N_3 is a constant independent of x, t_2, t_3.*

In a way similar to the calculations of (5.19), we have

$$\begin{aligned}&\partial_x \ln v = -N_1 - \partial_x \ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}} \qquad \partial_{t_2}\ln v = -\hat N_2 - \partial_{t_2}\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}}\\ &\partial_{t_3}\ln v = -\hat N_3 - \partial_{t_3}\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}}\end{aligned} \tag{5.25}$$

where $\hat N_2$ is a constant independent of x, t_2, t_3, t_4 and $\hat N_3$ is a constant independent of x, t_2, t_3:

$$\begin{aligned}\hat N_2 &= -\frac{1}{2}\big(\partial_{t_2} + \partial_x^2\big)\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}} - \partial_x^2\ln\theta_1^{(2)} - \frac{1}{2}\left(I_1 + \partial_x\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}}\right)^2 + \gamma_1\left(I_1 + \partial_x\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}}\right) + \frac{1}{2}I_2 - \gamma_2\\ \hat N_3 &= -\frac{2}{3}\left(\partial_{t_3} - \frac{1}{4}\partial_x^3 + \frac{3}{4}\partial_x\partial_{t_2}\right)\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}} - \partial_x\partial_{t_2}\ln\theta_1^{(2)} + \gamma_2\left(I_1 + \partial_x\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}}\right)\\ &\quad - \frac{1}{2}\left(N_1 + \partial_x\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}}\right)\left[I_2 + \big(\partial_{t_2} - \partial_x^2\big)\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}} - 2\partial_x^2\ln\theta_1^{(2)}\right]\\ &\quad + \frac{1}{6}\left(I_1 + \partial_x\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}}\right)^2\left(I_1 - 3\gamma_1 + \partial_x\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}}\right) + \frac{1}{3}I_3 - \gamma_3.\end{aligned} \tag{5.26}$$

With the aid of (5.21), we arrive at

$$uv = -\partial_x^2\ln\theta_1^{(1)} + \tfrac{1}{2}\big(\gamma_1^2 - 2\gamma_2 - N_2 + I_2\big). \tag{5.27}$$

Based on the above results and using (2.4), (2.10), (5.11), (5.19) and (5.27), we obtain the assertion.

Theorem 5.3. *The (2+1)-dimensional breaking soliton equation (1.1) and the coupled KP equation (1.2) has algebraic-geometrical solutions, respectively,*

$$\hat q(x, y, t) = \hat q(0, 0, 0)\exp(-N_1 x - \hat N_2 y - \hat N_3 t)\frac{\theta(\Omega^{(0)}x + \Omega^{(1)}y + \Omega^{(2)}t + \Lambda^{(1)})\theta(\Lambda^{(2)})}{\theta(\Omega^{(0)}x + \Omega^{(1)}y + \Omega^{(2)}t + \Lambda^{(2)})\theta(\Lambda^{(1)})}$$

$$\hat{r}(x,y,t) = \hat{r}(0,0,0)\exp(N_1x + N_2y + N_3t)\frac{\theta(\Omega^{(0)}x + \Omega^{(1)}y + \Omega^{(2)}t + \Upsilon^{(2)})\theta(\Upsilon^{(1)})}{\theta(\Omega^{(0)}x + \Omega^{(1)}y + \Omega^{(2)}t + \Upsilon^{(1)})\theta(\Upsilon^{(2)})} \tag{5.28}$$

and

$$\begin{aligned} q(x,y,t) &= -4\partial_x^2 \ln\theta(\Omega^{(0)}x + \Omega^{(1)}y + \Omega^{(2)}t + \Upsilon^{(1)}) + a_0 \\ p(x,y,t) &= p(0,0,0)\exp(2N_1x + 2N_2y + 2N_3t)\frac{\theta^2(\Omega^{(0)}x + \Omega^{(1)}y + \Omega^{(2)}t + \Upsilon^{(2)})\theta^2(\Upsilon^{(1)})}{\theta^2(\Omega^{(0)}x + \Omega^{(1)}y + \Omega^{(2)}t + \Upsilon^{(1)})\theta^2(\Upsilon^{(2)})} \\ r(x,y,t) &= r(0,0,0)\exp(-2N_1x - 2\hat{N}_2y - 2\hat{N}_3t)\frac{\theta^2(\Omega^{(0)}x + \Omega^{(1)}y + \Omega^{(2)}t + \Lambda^{(1)})\theta^2(\Lambda^{(2)})}{\theta^2(\Omega^{(0)}x + \Omega^{(1)}y + \Omega^{(2)}t + \Lambda^{(2)})\theta^2(\Lambda^{(1)})} \end{aligned} \tag{5.29}$$

where $a_0 = 2\left(\gamma_1^2 - 2\gamma_2 - N_2 + I_2\right)$ *is a constant. And the first expression of (5.29) is also an algebraic-geometrical solution of the KP equation [10, 21]*

$$q_t = \tfrac{1}{4}\left(q_{xxx} - 3qq_x + 3\partial_x^{-1}q_{yy}\right).$$

Therefore, noting (2.12), (5.12) and (5.27), we have the following fact.

Theorem 5.4. *The (3+1)-dimensional nonlinear evolution equation (1.3) has algebraic-geometrical solutions*

$$w(x,y,t,z) = -3\partial_x^2 \ln\theta(\Omega^{(0)}x + \Omega^{(1)}y + \Omega^{(2)}t + \Omega^{(3)}z + \Upsilon^{(1)}) + b_0 \tag{5.30}$$

where $b_0 = \frac{3}{2}\left(\gamma_1^2 - 2\gamma_2 - N_2 + I_2\right)$ *is a constant.*

6. Conclusions

In the foregoing sections we have derived algebraic-geometrical solutions of the (2+1)-dimensional breaking soliton equation, the coupled KP equation with three potentials and the (3+1)-dimensional evolution equation, which was not considered in the literature before. Generally, it is very difficult for a given (2+1)-dimensional nonlinear evolution equation to be decomposed into two (1+1)-dimensional soliton equations in the same hierarchy. Here we split successfully the (2+1)-dimensional breaking soliton equation and the coupled KP equation with three potentials into two (1+1)-dimensional AKNS equations with the help of a direct way and the nonlinearization of a Lax pair. What is more significant is that this suggests a new possible approach to decompose multidimensional evolution equations and to construct their algebraic-geometrical solutions. A (3+1)-dimensional nonlinear evolution equation is cited as an instance in illustration of our method. These multidimensional equations are further decomposed into solvable ordinary differential equations by utilizing (1+1)-dimensional AKNS equations. Based on the decomposition and the theory of algebraic curve, the explicit solutions of these multidimensional flows are expressed simply in a way of linear superposition by the introduced Abel–Jacobi coordinates. An inverse procedure is discussed to transform the explicit solutions in the original coordinates, from which algebraic-geometrical solutions of these multidimensional nonlinear evolution equations are given, whose expressions are very brief.

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